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CONFORMAL MAPPINGS OF FINITE RIEMANN SURFACES

ABSTRACT

The purpose of this work is to obtain a new type of conformal mappings of compact finite Riemann surfaces bounded by finitely many analytic Jordan curves. This is achieved by making use of Riemann-Roch theorem. As is well-known, every plane region is conformally equivalent to a parallel slit region. This theorem was carried over the case of Riemann surfaces with finite genus. The other types of conformal mappings can be found in the different literatures. It will be now deal with a different conformal mapping from those. It is a finite sheeted covering surface of the extended complex plane whose each boundary component consists of a closed interval on real axis.
Keywords: Riemann Surfaces, Conformal Mapping, Jordan Curves, Riemann-Roch Theorem, Meromorphic Function

SONLU RIEMANN YÜZEYLERİNİN KONFORMAL DÖNÜŞÜMLERİ

ÖZET

Bu çalışmanın amacı, sonlu sayıda analitik Jordan eğrileri ile sınırlanan kompakt sonlu Riemann yüzeylerinin konformal dönüşümlerinin yeni bir tipini elde etmektir. Bu, Riemann-Roch teoreminden yararlanılarak elde edilir. Bilindiği üzere, her bir düzlem bölgesi konformal olarak paralel bir yarık bölgeye eşdeğerdir. Bu teorem cinsi sonlu Riemann yüzeylerine uygulanmıştır. Konformal dönüşümlerin diğer tipleri farklı eserlerde incelenmiştir. Burada farklı bir konformal dönüşümle ilgilenilecektir. Her sınır bileşeni gerçel eksen üzerinde ve kapalı bir aralık genişletilmiş kompleks düzlemin sonlu örtüsüdür.
Anahtar Kelimeler: Riemann Yüzeyleri, Konformal Dönüşüm, Jordan Eğrileri, Riemann-Roch Teoremi, Meromorfik Fonksiyonu

1. INTRODUCTION (GİRİŞ)

Let W be an open Riemann surfaces of genus g. A Lebesque measurable complex differential $\lambda = a(z)dx + b(z)dy$ on W is said to be square integrable, if the integral $\iint_W \left(|a|^2 + |b|^2\right)dxdy$ is finite. The totality of square integrable complex differentials on W forms a real Hilbert space $\Lambda = \Lambda(W)$ with the usual inner product defined by

$$\langle \lambda_1, \lambda_2 \rangle = Re(\lambda_1, \lambda_2) = Re \iint_W \lambda_1 \wedge \overline{\lambda_2^*} = Re \iint_W \left(a_1 \overline{a_2} + b_1 \overline{b_2}\right)dxdy$$

where $\lambda_j = a_j(z)dx + b_j(z)dy$, ($j$=1, 2) for a local parameter $z = x + iy$. We will denote the complex conjugate of λ by $\overline{\lambda}$ and the star conjugate differential of λ by λ^*.

Λ_h, Λ_{hse} and Λ_{eo} stand for the real Hilbert spaces of complex square integrable differentials on W with some restricted properties. Λ_c is the real Hilbert space of complex square integrable closed differentials on W.

We now give a Lemma that we often use in this paper.

- **Lemma 1.1. (Yardımcı Teorem 1.1.)**

Let G be a canonical regular region on W and $\Xi(W) = \{A_j, B_j\}_{j=1}^{g}$ be a canonical homology basis on W modulo dividing cycles such that $\Xi \cap \overline{G}$ forms a canonical homology basis on $\overline{G}$ modulo ∂G. Suppose λ_1 and λ_2 are closed C^1 - differentials on G and λ_1 is semiexact, then

$$\langle \lambda_1, \lambda_2^* \rangle_G = -Re \int_{\partial G} (\int \lambda_1) \overline{\lambda_2} + \sum_j Re(\int_{A_j} \lambda_1 \int_{B_j} \overline{\lambda_2} - \int_{B_j} \lambda_1 \int_{A_j} \overline{\lambda_2})$$

where $\sum_j$ stands for the sum over all A_j, B_j contained in G (cf. [1-4]).

(The meaning of the $\int \lambda_1$ is the following: We cut $\overline{G}$ along A_j, B_j to make it a planar surface $\overline{G}_0$. Since λ_1 is semiexact, there exists a C^2 - function f on G_0 such that df $= \lambda_1$ and consider the integral $\int \lambda_1$ on ∂G_0).

- **Definition 1.1. (Tanım 1.1.)**

A closed subspace Λ_0 of $\Lambda_{hse}(W)$ is called *behavior space* if it satisfies the following conditions.

(i) $\Lambda_0 = i\Lambda_0^{*\perp}$, where $\Lambda_0^{\perp}$ is the orthogonal complement in Λ_h of Λ_0.

(ii) For each $\lambda \in \Lambda_0$, $\int_{A_j} \lambda = 0$, for $j = 1,2,...,g$.

It is now an easy matter to verify that $\overline{\Lambda_0} = \{\lambda \in \Lambda_h : \bar{\lambda} \in \Lambda_0\}$ is also a behavior space if Λ_0 is a behavior space.

- **Definition 1.2. (Tanım 1.2.)**

A meromorphic differential λ on *W* will be said to have Λ_0-behavior if there exists a neighborhood *U* of the ideal boundary of *W* on which it can be written as $\lambda = \lambda_0 + \lambda_{eo}$ where $\lambda_0 \in \Lambda_0$ and $\lambda_{eo} \in \Lambda_{eo} \cap \Lambda^1$.

A meromorphic function *f* on *W* is said to have Λ_0-*behavior* if *df* has Λ_0-behavior.

- **Definition 1.3. (Tanım 1.3.)**

Two behavior spaces Λ_0^1 and Λ_0^2 are called *dual to each other* if and only if
$\left(\lambda_1, \overline{\lambda_2^*}\right) \in \mathbf{R}$ for $\lambda_1 \in \Lambda_0^1$ and $\lambda_2 \in \Lambda_0^2$

- **Lemma 1.2. (Yardımcı Teorem 1.2.)**

The behavior spaces Λ_0 and $\overline{\Lambda_0}$ are dual to each other.

- **Proof (İspat)**

If Λ_0 is a behavior space, so is $\overline{\Lambda_0} = \{\lambda \in \Lambda_h : \bar{\lambda} \in \Lambda_0\}$. Definition 1.3 can be easily verified, for $\lambda_1 \in \Lambda_0$ and $\lambda_2 \in \overline{\Lambda_0}$

$$(\lambda_1, (\overline{\overline{\lambda_2^*}})) = Re(\lambda_1, (\overline{\overline{\lambda_2^*}})) + i\,Im(\lambda_1, (\overline{\overline{\lambda_2^*}}))$$

$$= Re(\lambda_1, \lambda_2^*) + i\,Im(\lambda_1, \lambda_2^*)$$

$$= Re(\lambda_1, \lambda_2^*) + i\,Re(\lambda_1, i\lambda_2^*)$$

$$= \langle \lambda_1, \lambda_2^* \rangle + i \langle \lambda_1, i\lambda_2^* \rangle$$

for every $\lambda_1 \in \Lambda_0$ and $\lambda_2 \in \Lambda_0$. Since Λ_0 *is a behavior space, we conclude that*

$$\langle \lambda_1, i\lambda_2^* \rangle = 0$$

and

$$(\lambda_1, (\overline{\overline{\lambda_2^*}}) \in \mathbf{R}$$

Which proves the behavior spaces Λ_0 *and* $\overline{\Lambda_0}$ *are dual to each other* Let *W* be an open Riemann surface of genus g $(0 < g < \infty)$ and δ be a finite divisor on *W. We consider the following sets which evidently from linear spaces over* **R**.

$$S(\Lambda_0^1,\partial^{-1})=\left\{ f: \begin{array}{l} \text{(i) f is a single valued meromorphic function on W} \\ \text{(ii) df has } \Lambda_0^1-\text{behavior} \\ \text{(iii) The divisor of f is multiple of } \partial^{-1} \end{array} \right\}$$

$$D(\Lambda_0^2,\ \partial)=\left\{ \lambda: \begin{array}{l} \text{(i) } \lambda \text{ is a meromorphic differential on W} \\ \text{(ii) } \lambda \text{ has } \Lambda_0^2-\text{behavior} \\ \text{(iii) The divisor of } \lambda \text{ is multiple of } \partial \end{array} \right\} [5]$$

- **Theorem 1.1. (Teorem 1.1.)**

(Riemann-Roch) Suppose that Λ_0^1 and Λ_0^2-behavior is dual to each other.

Let ∂ be a finite divisor on *W*. Then

$dim\, S(\Lambda_0^1,\partial^{-1}) = dim\, D(\Lambda_0^2,\partial)+2(deg\,\partial - g+1)$ [4-6]

2. REASERCH SIGNIFICANCE (ÇALIŞMANIN ÖNEMİ)

For years, on the conformal mappings of compact finite Riemann surfaces bounded by finitely many analytic Jordan curves have been studied. In this study, it is studied on a new type of these conformal mappings.

3. PART (3.BÖLÜM)

Let W be the interior a finite Riemann surface $\overline{W}$ *of genus g with h boundary components. Suppose that* $\partial W=\bigcup_{k=1}^{h}\beta_k$ *,* β_k *being a contour.* $Let\,\Xi=\{A_j,B_j\}_{j=1}^{g}$ be a canonical homology basis of $\overline{W}$ *modulo the border.* $\Lambda_c^1(\overline{W})$ *stands for the closed differentials on the border. We define*

$$\Lambda_q^1(\overline{W})=\left\{ \lambda\in\Lambda_c^1(\overline{W}): \begin{array}{l} \text{(i) } \lambda \text{ is semiexact,i.e., } \int_{\beta_k}\lambda=0 \text{ for all } \beta_k\ (1\le k\le h) \\ \text{(ii) } \int_{A_j}\lambda=0 \text{ for every } j=1,2,...,g \\ \text{(iii) } \lambda \text{ is real valued along } \beta_k\ (1\le k\le h)\text{,i.e.,a complex} \\ \text{differential } \lambda=a(z)dx+b(z)dy \text{ is said to be real valued} \\ \text{along } \beta_k \text{ ıf a } a(z(t))x'(t)+b(z(t))y'(t)\in R \text{ for all } t\in[0,1] \end{array} \right\}$$

- **Lemma 3.1. (Yardımcı Teorem 3.1.)**

$\Lambda_q^1(\overline{W})$ is the orthogonal complement of $i\Lambda_q^1(\overline{W})^*$ in $\Lambda_c^1(\overline{W})$, *that is,*

$\Lambda_q^1(\overline{W}) = i\Lambda_q^1(\overline{W})^{*\perp}$.

- **Proof (İspat)**

First we shall show that $\Lambda_q^1(\overline{W}) \perp i\Lambda_q^1(\overline{W})^*$. Take $\lambda_q \in \Lambda_q^1(\overline{W})$ and $\lambda \in \Lambda_q^1(\overline{W})$. Then by Lemma 1.1, we have

$$< \lambda_q, \lambda^* > = Re(\lambda_q, \lambda^*) = -\sum_{k=1}^{h} Re \int_{\beta_k} \left(\int \lambda_q \right) \overline{\lambda} + \sum_{j=1}^{g} Re \left(\int_{Aj} \lambda_q \int_{B_j} \overline{\lambda} - \int_{B_j} \lambda_q \int_{A_j} \overline{\lambda} \right)$$

$$= -\sum_{k=1}^{h} Re \int_{\beta_k} f_k \overline{\lambda} + \sum_{j=1}^{g} Re \left(\int_{Aj} \lambda_q \int_{B_j} \overline{\lambda} - \int_{B_j} \lambda_q \int_{A_j} \overline{\lambda} \right) \tag{1}$$

provided that $df_k = \lambda_q$ near $\beta_k (1 \le k \le h)$ Because of the semiexactness of λ_q we can take functions f_k seperately on each boundary component. Since $\int_{A_j} \lambda_q = 0$ and $\int_{A_j} \overline{\lambda} = 0$, the last term of equation (1) is zero.

From the definition of the $i\Lambda_q^1(\overline{W})$ the differential λ is i**R**-valued alongs $\beta_k (1 \le k \le h)$, that is, $Re(\lambda)$ is zero along β_k. Similarly, we know that f_k is real valued along β_k, the imaginary part of f_k is constand along β_k. $Im(f_k)$ is zero along $\beta_k (1 \le k \le h)$. Hence

$$Re \int_{\beta_k} f_k \lambda = \int_{\beta_k} Re(f_k).Re(\lambda) + \int_{\beta_k} Im(f_k).Im(\lambda) = 0$$

Since these properties are valid for all $k, j (1 \le k \le h, 1 \le j \le g)$ It follows that

$$< \lambda_q, \lambda^* > = 0$$

Next we prove the converse of lemma. That is ıf it is $< \lambda_q, \lambda^* > = 0$ for each $\lambda_q \in \Lambda_q^1(\overline{W})$ then λ is in the space $i\Lambda_q^1(\overline{W})$.

Now we can construct a semiexect C^1-differential $\lambda_0 = \lambda_{k_0 j_0}(U_{k_0}, c_{k_0}, C_{k_0})$ $1 \le k_0 \le h$, $1 \le j_0 \le g$ such that

(i) $$\int \lambda_0 = \begin{cases} u_{k_0} + c_{k_0}, & on \beta_{k_0} \\ 0, & on \beta_k (k \ne k_0) \end{cases}$$

$$\int_{A_{j_0}} \lambda_0 = 0 \quad , \quad \int_{B_{j_0}} \lambda_0 = C_{j_0}$$

(ii)

$$\int_{A_j, B_j} \lambda_0 = 0 \quad , \quad for \quad j \ne j_0$$

where

$u_{k_0} \in C_G^2(\beta_{k_0})$={all the real valued twice continuously differentiable functions defined on β_{k_0}}and $C_{k_0} \in$**R** or $C_{k_0} \in$i**R** and $C_{j_0} \in$**C**.(In the integral $\int \lambda_0$ is understood in the sense of Lemma 1.1)

Such a differantial is obtained by a standart method as follows:

Let G be a relatively compact ring domain containing A_{j_0} which may be assumed to be an orientable analytic Jordan curve. For any $u_{k_0} \in C_G^2(\beta_{k_0})$ and c_{k_0}, C_{j_0} we take a function F defined on $G \cup \beta$ such that

$$F = \begin{cases} u_{k_0} + c_{k_0}, & n.b.d \ of \ \beta_{k_0} \\ C_{j_0}, & on \ the \ right \ part \ of \ G \\ 0, & else \ where \end{cases}$$

We can extended F so as F belongs to $C^2(\overline{W} - A_{j_0})$. If we set $\lambda_0 = dF$, λ_0 is the desired differential.

Now suppose that $<\lambda_q, \lambda^*> = 0$ for all $\lambda_q \in \Lambda_q^1(\overline{W})$. By Lemma 1.1

$$\sum_k Re \int_{\beta_k} \left(\int \lambda_q\right)\overline{\lambda} + \sum_{j=1}^{g} Re\left(\int_{Aj} \lambda_q \int_{B_j} \overline{\lambda} - \int_{B_j} \lambda_q \int_{A_j} \overline{\lambda}\right) = 0 \qquad (2)$$

In equation (2) we can take $\lambda_q = \lambda_{k_0 j_0}(0,1,0)$ and $\lambda_q = \lambda_{k_0 j_0}(0,i,0)$ as λ_q we obtain that

$$Re \int_{\beta_{k_0}} \overline{\lambda} = Re \int_{\beta_{k_0}} i \ \overline{\lambda} = 0$$

and hence

$$\int_{\beta_{k_0}} \lambda = 0$$

which proves the semiexacness of λ

In equation(2) setting $\lambda_q = \lambda_{k_0 j_0}(u_{k_0}, 0, 0)$ we conclude that

$$Re \int_{\beta_{k_0}} u_{k_0} \overline{\lambda} = 0$$

This holds for all $u_{k_0} \in C_G^2(\beta_{k_0})$, and therefore we can conclude that λ is i**R**-valued along β_{k_0}.

In equation (2), finally we set $\lambda_q = \lambda_{k_0 j_0}(0,0,1)$ and $\lambda_q = \lambda_{k_0 j_0}(0,0,i)$. Then it follows that

$$Re \int_{A_{j_0}} \overline{\lambda} = Re\left[i \int_{A_{j_0}} \overline{\lambda}\right] = 0$$

Therefore, we have

$$\int_{A_{j_0}} \lambda = 0$$

Since these results are valid for all k_0 and j_0 ($1 \le k_0 \le h$, $1 \le j_0 \le g$) we can conclude that $\lambda \in i\Lambda_q^1(\overline{W})$

4. PART (4. BÖLÜM)

Now we take the space of the complex harmonic differentials $\Lambda_h(\overline{W})$ instead of $\Lambda_c^1(\overline{W})$ and the subspace $\Lambda_0(\overline{W}) \subset \Lambda_h(\overline{W})$ instead of $\Lambda_q^1(\overline{W})$. By Defination 3.1 we have

$$\Lambda_0(\overline{W}) = \Lambda_q^1(\overline{W}) \cap \Lambda_h(\overline{W})$$

and

$$i\Lambda_0(\overline{W}) = (i\Lambda_q^1(\overline{W})) \cap \Lambda_h(\overline{W})$$

Also, the subspaces $\Lambda_0(\overline{W})$ and $i\Lambda_0(\overline{W})$ are closed in Λ_h. By Lemma 3.1 we have

$$\Lambda_0(\overline{W}) = i\Lambda_0(\overline{W})^{*\perp}$$

(where $\Lambda^\perp$ is the orthogonal complement in Λ_h), and

$$\Lambda_h = \Lambda_0(\overline{W}) \oplus i\Lambda_0(\overline{W})^* .$$

- **Theorem 4.1. (Teorem 4.1.)**

Let *W* be the interior of finite Riemann surface of genus *g*. Suppose that $\beta = \beta(W)$, the border of W, consists of bordered of *h* components $\beta_1, \beta_2, \ldots, \beta_h$. We shall use $\overline{W}$ to denote $W \cup \beta(W)$. Then there exists a meromorphic function f on W such that

(i) f maps each β_h $(1 \le k \le h)$ to a closed interval on the real axis.

(ii) f maps some of $g+1$ preassigned points on W to the point at infinity.

(iii) $f(W)$, the image of W under f, is at most $(g+1)$-sheeted over the extended complex plane.

- **Proof (İspat)**

The subspace $\Lambda_0(\overline{W})$ satisfies all the conditions in Definition 1.1. Therefore $\Lambda_0(\overline{W})$ is a behavior space. Hence by Lemma 1.2 we know that $\Lambda_0(\overline{W})$ and $\overline{\Lambda}(\overline{W})$ define dual behaviors with respect to **R.** Riemann-Roch theorem is now applicable for these boundary behaviors and we know that there exists a non-constant meromorphic function f with $\Lambda_0(\overline{W})$- behavior possible $(g+1)$ points P_r $(0 \le r \le g)$ on W. Indeed, Theorem 1.1 gives the following result

$$dim\, S(\Lambda_0(\overline{W}), \delta^{-1}) = dim\, D(\overline{\Lambda_0}(\overline{W}), \delta) + 2(deg\, \delta - g + 1)$$

$$\ge 2(deg\, \delta - g + 1)$$

If we set $\delta = P_0 P_1 \ldots P_g$ then $deg\, \delta = g+1$ and $dim\, S(\Lambda_0(\overline{W}), \delta^{-1}) \ge 2$.

Therefore there exists a non-constant meromorphic function f such that the function f has $\Lambda_0(\overline{W})$- behavior and so df is real-valued along β_k, that is $Im(f)$ is constant on β_k. Thus by using of the argument principle in [7], we conclude that $f(W)$, the image of W under f, is at most $(g+1)$-sheeted over the extended complex plane.

Remarks (1) Instead of $\Lambda_0(\overline{W})$ we take the space Λ_{hm} of the square integrable semiexact complex differentials on $\overline{W}$ whose the real part is harmonic measure. Thus these are canonical semiexact differentials in equation (1), provided that the genus of W is zero. Also the function f mentioned in Theorem 4.1 belongs to the class $\Re_0$ equation (1).

5. CONCLUSION (SONUÇ)

In Başkan’s works [4 and 5], we take Λ_B-behavior space instead of $\Lambda_0(\overline{W})$. Generalization of Theorem 3 in Shiba’s work [2] can be obtained by using generalized divisors. Theorem 3.1 can also be considered a generalization of Theorem 2 in Shiba’s work [2].

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